

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i_1, i_2, \dots, i_k=1}^n \lambda_1^{|i_1 - i_2 - s_1|} \lambda_2^{|i_2 - i_3 - s_2|} \dots \lambda_k^{|i_k - i_1 - s_k|}$$

Formula to evaluate

Yuhao Liu¹ and Jan Vrbik¹

¹*Department of Mathematics and Statistics, Brock University,
Canada*

June 11, 2015

Abstract

Computing moments of various parameter estimators related to an autoregressive model of Statistics, one needs to evaluate several expressions of the type mentioned in the title of this article. We proceed to derive the corresponding formulas.

1 Introduction

The autoregressive model of Statistics generates a random sequence of observations by

$$X_i = \alpha_1 X_{i-1} + \alpha_2 X_{i-2} + \dots + \alpha_k X_{i-k} + \varepsilon_i \quad (1)$$

where ε_i are independent, Normally distributed random variables with the mean of 0 and the same standard deviation, and k is a fixed integer, usually quite small (e.g. $k = 1$ defines the so called Markov model). The sufficient and necessary condition for the resulting sequence to be asymptotically stationary is that all k solutions of the characteristic polynomial

$$\lambda^k = \alpha_1 \lambda^{k-1} + \alpha_2 \lambda^{k-2} + \dots + \alpha_k \quad (2)$$

are, in absolute value, smaller than 1 (this is then assumed from now on).

The j th-order serial correlation coefficient ρ_j (between X_i and X_{i+j}) is then computed by

$$\rho_j = A_1 \lambda_1^{|j|} + A_2 \lambda_2^{|j|} + \dots + A_k \lambda_k^{|j|} \quad (3)$$

where the λ_i 's are the k roots of (2), and the A_i coefficients are themselves simple functions of these roots.

Computing the first few moments of various estimators (of the α_i parameters) boils down to computing moments of expressions of the

$$\sum_{i=1}^n X_i \quad (4)$$

and

$$\sum_{i=1}^{n-j} X_i X_{i+j} \quad (5)$$

type, where X_1, X_2, \dots, X_n is a collection of n consecutive observations (assuming that the process has already reached its stationary phase).

This in turn requires evaluating various summations (see [1]), of which the most difficult are

$$\sum_{i_1, i_2=1}^{\tilde{n}} \lambda_1^{|i_1-i_2+s_1|} \lambda_2^{|i_2-i_1+s_2|} \quad (6)$$

$$\sum_{i_1, i_2, i_3=1}^{\tilde{n}} \lambda_1^{|i_1-i_2+s_1|} \lambda_2^{|i_2-i_3+s_2|} \lambda_3^{|i_3-i_1+s_3|} \quad (7)$$

and

$$\sum_{i_1, i_2, i_3, i_4=1}^{\tilde{n}} \lambda_1^{|i_1-i_2+s_1|} \lambda_2^{|i_2-i_3+s_2|} \lambda_3^{|i_3-i_4+s_3|} \lambda_4^{|i_4-i_1+s_4|} \quad (8)$$

where $\lambda_1, \lambda_2, \lambda_3$, and λ_4 are any of the λ_i roots (some may appear in duplicate), s_1, s_2, s_3 , and s_4 are small integers, and \tilde{n} indicates that each of the upper limits equals to n , perhaps adjusted in the manner of (5).

It is possible (but rather messy — the result depends on the values of s_1, s_2 and \tilde{n} — see [2]) to *exactly* evaluate (6) and realize that the answer will always (this goes for the other two summations as well) consist of three parts:

1. terms proportional to λ_i^n , which tend to zero (as n increases) ‘exponentially’,
2. terms which stay constant as n increases,
3. terms proportional to n .

Luckily, to build the approximation which is usually deemed sufficient (see [1]), we need to find only the n proportional terms. These can be extracted by dividing the relevant summation by n and taking the $n \rightarrow \infty$ limit. Incidentally, this results in the following (and most welcomed) simplification: the corresponding answer will be the *same* regardless of the \tilde{n} adjustments (thus, we may as well use n instead), and will similarly *not* depend on the individual s_i ’s, but only on the *absolute* value of their *sum*. The proof of this statement is omitted.

2 Evaluating the limits

Starting with (6), we obtain

$$\begin{aligned}
F_2(\lambda_1, \lambda_2; S) & \equiv \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i_1, i_2=1}^{\tilde{n}} \lambda_1^{|i_1 - i_2 + s_1|} \lambda_2^{|i_2 - i_1 + s_2|} \\
& = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i_1, i_2=1}^n \lambda_1^{|i_1 - i_2|} \lambda_2^{|i_2 - i_1 + S|} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=-n}^n \sum_{i_1=\max(1, 1-j)}^{\min(n, n-j)} \lambda_1^{|j|} \lambda_2^{|S-j|} \\
& = \lim_{n \rightarrow \infty} \sum_{j=-n}^n \frac{n - |j|}{n} \lambda_1^{|j|} \lambda_2^{|S-j|} = \sum_{j=-\infty}^{\infty} \lambda_1^{|j|} \lambda_2^{|S-j|} \\
& = \sum_{j=-\infty}^0 \lambda_1^{-j} \lambda_2^{S-j} + \sum_{j=1}^S \lambda_1^j \lambda_2^{S-j} + \sum_{j=S+1}^{\infty} \lambda_1^j \lambda_2^{j-S} \\
& = \frac{\lambda_1^{S+1}(1 - \lambda_2^2)}{(\lambda_1 - \lambda_2)(1 - \lambda_1 \lambda_2)} + \frac{\lambda_2^{S+1}(1 - \lambda_1^2)}{(\lambda_2 - \lambda_1)(1 - \lambda_2 \lambda_1)}
\end{aligned}$$

where $S \equiv |s_1 + s_2|$. Following the usual convention, an empty summation (such as $\sum_{j=1}^0$) has a zero value.

Note that the answer can be written in the following form:

$$\sum_{i=1}^{\ell} \lambda_i^{S+\ell-1} \prod_{j=1, j \neq i}^{\ell} \frac{1 - \lambda_j^2}{(\lambda_i - \lambda_j)(1 - \lambda_i \lambda_j)} \quad (9)$$

with $\ell = 2$. Also note that, when $\lambda_2 = \lambda_1$, the value of $F_2(\lambda_1, \lambda_1; S)$ can be easily obtained by

$$\lim_{\lambda_2 \rightarrow \lambda_1} F_2(\lambda_1, \lambda_2; S) = \frac{\lambda_1^S (1 + S + (1 - S)\lambda_1^2)}{1 - \lambda_1^2}$$

2.1 The case of 3 λ 's

Moving on to (7), we now get

$$\begin{aligned}
F_3(\lambda_1, \lambda_2, \lambda_3; S) &\equiv \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i_1, i_2, i_3=1}^{\tilde{n}} \lambda_1^{|i_1-i_2+s_1|} \lambda_2^{|i_2-i_3+s_2|} \lambda_3^{|i_3-i_1+s_3|} \\
&= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i_1, i_2, i_3=1}^n \lambda_1^{|i_1-i_2|} \lambda_2^{|i_2-i_3|} \lambda_3^{|i_3-i_1+S|} \\
&= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j_1=-n}^n \sum_{j_2=\max(-n, -n-j_1)}^{\min(n, n-j_1)} \sum_{i_1=\max(1, 1-j_2, 1-j_1-j_2)}^{\min(n, n-j_2, n-j_1-j_2)} \lambda_1^{|j_1|} \lambda_2^{|j_2|} \lambda_3^{|S-j_1-j_2|} \\
&= \lim_{n \rightarrow \infty} \sum_{j_1=-n}^n \sum_{j_2=\max(-n, -n-j_1)}^{\min(n, n-j_1)} \frac{\min(n, n-j_2, n-j_1-j_2) - \max(1, 1-j_2, 1-j_1-j_2) + 1}{n} \cdot \lambda_1^{|j_1|} \lambda_2^{|j_2|} \lambda_3^{|S-j_1-j_2|} \\
&= \sum_{j_1=-\infty}^{\infty} \sum_{j_2=-\infty}^{\infty} \lambda_1^{|j_1|} \lambda_2^{|j_2|} \lambda_3^{|S-j_1-j_2|}
\end{aligned}$$

where $S \equiv |s_1 + s_2 + s_3|$.

This time, evaluating the last summation is slightly more difficult; we will do it quadrant by quadrant.

For the first quadrant (*including* the adjacent half-axes and the origin), we get (visualize the quadrant, cut by the $S = j_1 + j_2$ line):

$$\begin{aligned}
&\sum_{j_1=0}^S \sum_{j_2=0}^{S-j_1} \lambda_1^{j_1} \lambda_2^{j_2} \lambda_3^{S-j_1-j_2} + \sum_{j_1=0}^S \sum_{j_2=S-j_1+1}^{\infty} \lambda_1^{j_1} \lambda_2^{j_2} \lambda_3^{j_1+j_2-S} + \sum_{j_1=S+1}^{\infty} \sum_{j_2=0}^{\infty} \lambda_1^{j_1} \lambda_2^{j_2} \lambda_3^{j_1+j_2-S} \\
&= \frac{\lambda_1^{S+2}(1-\lambda_3^2)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(1-\lambda_1\lambda_3)} + \frac{\lambda_2^{S+2}(1-\lambda_3^2)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(1-\lambda_2\lambda_3)} + \frac{\lambda_3^{S+2}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}
\end{aligned}$$

For the second quadrant (again, *including* the corresponding boundaries - the resulting duplication with the first quadrant will be removed later), the same kind of approach yields

$$\begin{aligned}
&\sum_{j_1=-\infty}^0 \sum_{j_2=0}^{S-j_1} \lambda_1^{-j_1} \lambda_2^{j_2} \lambda_3^{S-j_1-j_2} + \sum_{j_1=-\infty}^0 \sum_{j_2=S-j_1+1}^{\infty} \lambda_1^{-j_1} \lambda_2^{j_2} \lambda_3^{j_1+j_2-S} \\
&= \frac{\lambda_2^{S+1}(1-\lambda_3^2)}{(\lambda_2-\lambda_3)(1-\lambda_2\lambda_1)(1-\lambda_2\lambda_3)} + \frac{\lambda_3^{S+1}}{(\lambda_3-\lambda_2)(1-\lambda_3\lambda_1)}
\end{aligned}$$

The fourth quadrant clearly results in the same answer, with λ_1 and λ_2 interchanged, namely

$$\frac{\lambda_1^{S+1}(1-\lambda_3^2)}{(\lambda_1-\lambda_3)(1-\lambda_1\lambda_2)(1-\lambda_1\lambda_3)} + \frac{\lambda_3^{S+1}}{(\lambda_3-\lambda_1)(1-\lambda_3\lambda_2)}$$

Finally, the third quadrant (including its boundaries) contributes

$$\sum_{j_1=-\infty}^0 \sum_{j_2=-\infty}^0 \lambda_1^{-j_1} \lambda_2^{-j_2} \lambda_3^{S-j_1-j_2} = \frac{\lambda_3^S}{(1-\lambda_3\lambda_1)(1-\lambda_3\lambda_2)}.$$

Adding the four results does *not* yield the desired answer, since the contribution of each of the two axes has been included *twice*, and that of the origin altogether *four* times. This can be easily corrected by subtracting $F_2(\lambda_2, \lambda_3; S)$ which removes the extra contribution of the $j_1 = 0$ axis, and $F_2(\lambda_1, \lambda_3; S)$ which does the same thing with the $j_2 = 0$ terms. This leaves us with the origin ($j_1 = j_2 = 0$) which, at this point, is still contributing *double* its value (two contributions have been removed with the two axes); subtracting λ_3^S fixes that as well.

The final answer thus becomes

$$\begin{aligned} F_3(\lambda_1, \lambda_2, \lambda_3; S) = & \frac{\lambda_1^{S+2}(1-\lambda_2^2)(1-\lambda_3^2)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(1-\lambda_1\lambda_2)(1-\lambda_1\lambda_3)} \\ & + \frac{\lambda_2^{S+2}(1-\lambda_1^2)(1-\lambda_3^2)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(1-\lambda_2\lambda_1)(1-\lambda_2\lambda_3)} \\ & + \frac{\lambda_3^{S+2}(1-\lambda_1^2)(1-\lambda_2^2)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(1-\lambda_3\lambda_1)(1-\lambda_3\lambda_2)} \end{aligned}$$

Note that this has the form of (9) with $\ell = 3$.

When any of the three λ 's are identical, the answer can be found as the corresponding limit of the previous expression. Thus, for example

$$F_3(\lambda, \lambda, \lambda; S) = \frac{2 + 3S + S^2 + 2(4 - S^2)\lambda^2 + (2 - 3S + S^2)\lambda^4}{2(1 - \lambda^2)^2} \cdot \lambda^S$$

etc.

2.2 The case of 4 λ 's

The main challenge is to evaluate the last limit, namely

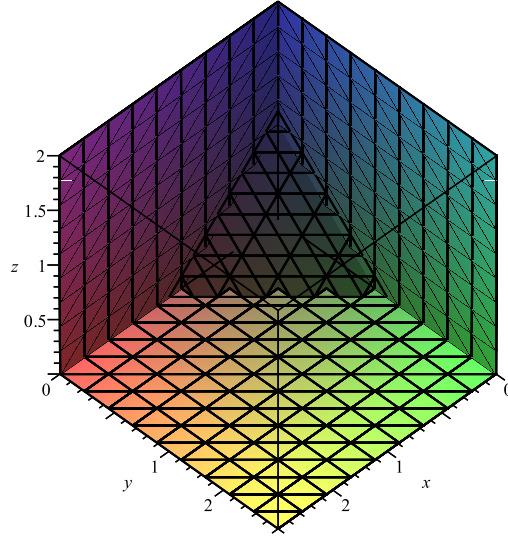
$$\begin{aligned} F_4(\lambda_1, \lambda_2, \lambda_3, \lambda_4; S) \\ \equiv \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i_1, i_2, i_3, i_4=1}^{\tilde{n}} \lambda_1^{|i_1-i_2+s_1|} \lambda_2^{|i_2-i_3+s_2|} \lambda_3^{|i_3-i_4+s_3|} \lambda_4^{|i_4-i_1+s_4|} \\ = \sum_{j_1=-\infty}^{\infty} \sum_{j_2=-\infty}^{\infty} \sum_{j_3=-\infty}^{\infty} \lambda_1^{|j_1|} \lambda_2^{|j_2|} \lambda_3^{|j_3|} \lambda_4^{|S-j_1-j_2-j_3|} \end{aligned}$$

We proceed octant by octant; the octants will be identified by the signs of the j_1 , j_2 and j_3 indices, respectively.

For the first octant denoted O_{+++} (including the adjacent portions of coordinate planes, axes and the origin), we get

$$\begin{aligned}
& \sum_{j_1=0}^S \sum_{j_2=0}^{S-j_1} \sum_{j_3=0}^{S-j_1-j_2} \lambda_1^{j_1} \lambda_2^{j_2} \lambda_3^{j_3} \lambda_4^{S-j_1-j_2-j_3} \\
& + \sum_{j_1=0}^S \sum_{j_2=0}^{S-j_1} \sum_{j_3=S-j_1-j_2+1}^{\infty} \lambda_1^{j_1} \lambda_2^{j_2} \lambda_3^{j_3} \lambda_4^{j_1+j_2+j_3-S} \\
& + \sum_{j_1=0}^S \sum_{j_2=S-j_1+1}^{\infty} \sum_{j_3=0}^{\infty} \lambda_1^{j_1} \lambda_2^{j_2} \lambda_3^{j_3} \lambda_4^{j_1+j_2+j_3-S} \\
& + \sum_{j_1=S+1}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} \lambda_1^{j_1} \lambda_2^{j_2} \lambda_3^{j_3} \lambda_4^{j_1+j_2+j_3-S} \\
& = \frac{\lambda_1^{S+3}(1-\lambda_4^2)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)(1-\lambda_1\lambda_4)} \\
& + \frac{\lambda_2^{S+3}(1-\lambda_4^2)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)(1-\lambda_2\lambda_4)} \\
& + \frac{\lambda_3^{S+3}(1-\lambda_4^2)}{(\lambda_3-\lambda_2)(\lambda_3-\lambda_1)(\lambda_3-\lambda_4)(1-\lambda_3\lambda_4)} \\
& + \frac{\lambda_4^{S+3}}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}
\end{aligned}$$

To understand why it was necessary to break the summation into four parts, it helps to visualize the first octant, cut by the $S = j_1 + j_2 + j_3$ plane, thus:



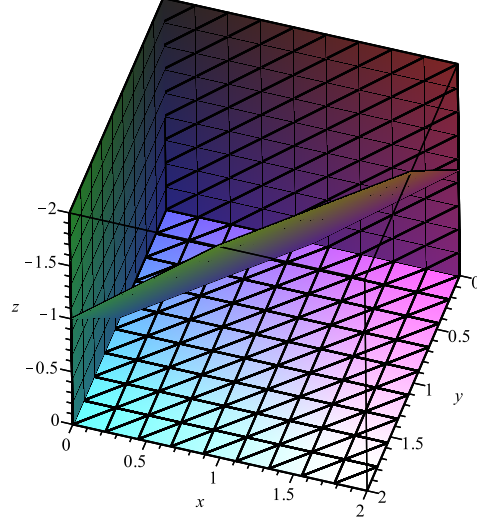
Our brain can interpret this image in two different ways; please make an effort to see the triangle as the most *distant* part of the picture.

As the next octant we take O_{++-} (with all its boundaries), contributing

$$\begin{aligned}
& \sum_{j_1=0}^S \sum_{j_2=0}^{S-j_1} \sum_{j_3=-\infty}^0 \lambda_1^{j_1} \lambda_2^{j_2} \lambda_3^{-j_3} \lambda_4^{S-j_1-j_2-j_3} \\
& + \sum_{j_1=0}^{\infty} \sum_{j_2=S-j_1+1}^{\infty} \sum_{j_3=-\infty}^{S-j_1-j_2} \lambda_1^{j_1} \lambda_2^{j_2} \lambda_3^{-j_3} \lambda_4^{S-j_1-j_2-j_3} \\
& + \sum_{j_1=0}^{\infty} \sum_{j_2=S-j_1+1}^{\infty} \sum_{j_3=S-j_1-j_2+1}^0 \lambda_1^{j_1} \lambda_2^{j_2} \lambda_3^{-j_3} \lambda_4^{j_1+j_2+j_3-S} \\
& + \sum_{j_1=S+1}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=-\infty}^{S-j_1-j_2} \lambda_1^{j_1} \lambda_2^{j_2} \lambda_3^{-j_3} \lambda_4^{S-j_1-j_2-j_3} \\
& + \sum_{j_1=S+1}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=S-j_1-j_2+1}^0 \lambda_1^{j_1} \lambda_2^{j_2} \lambda_3^{-j_3} \lambda_4^{j_1+j_2+j_3-S} \\
& = \frac{\lambda_1^{S+2}(1-\lambda_4^2)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_4)(1-\lambda_1\lambda_3)(1-\lambda_1\lambda_4)} \\
& + \frac{\lambda_2^{S+2}(1-\lambda_4^2)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_4)(1-\lambda_2\lambda_3)(1-\lambda_2\lambda_4)} \\
& + \frac{\lambda_4^{S+2}}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(1-\lambda_4\lambda_3)}
\end{aligned}$$

Again, visualizing the situation may help (the corner being the most *distant*

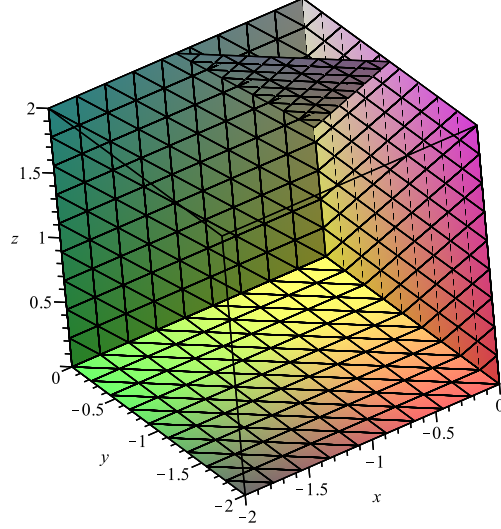
part of the picture):



The O_{-++} and O_{+-+} octants contribute the same expression each, after the $\lambda_3 \leftrightarrow \lambda_1$ and $\lambda_3 \leftrightarrow \lambda_2$ interchange, respectively.
 For O_{--+} (including boundaries) we get

$$\begin{aligned}
 & \sum_{j_1=-\infty}^0 \sum_{j_2=-\infty}^0 \sum_{j_3=0}^{S-j_1-j_2} \lambda_1^{-j_1} \lambda_2^{-j_2} \lambda_3^{j_3} \lambda_4^{S-j_1-j_2-j_3} \\
 & + \sum_{j_1=-\infty}^0 \sum_{j_2=-\infty}^0 \sum_{j_3=S-j_1-j_2+1}^{\infty} \lambda_1^{-j_1} \lambda_2^{-j_2} \lambda_3^{j_3} \lambda_4^{j_1+j_2+j_3-S} \\
 & = \frac{\lambda_3^{S+1}(1-\lambda_4^2)}{(\lambda_3-\lambda_4)(1-\lambda_3\lambda_1)(1-\lambda_3\lambda_2)(1-\lambda_3\lambda_4)} \\
 & + \frac{\lambda_4^{S+1}}{(\lambda_4-\lambda_3)(1-\lambda_4\lambda_1)(1-\lambda_4\lambda_2)}
 \end{aligned}$$

because this is how it looks like (again, the corner to be seen as most distant)



and similarly for O_{+-} and O_{-+-} , after the $\lambda_3 \leftrightarrow \lambda_1$ and $\lambda_3 \leftrightarrow \lambda_2$ interchange, respectively.

Finally, O_{--} with its boundaries contributes

$$\begin{aligned} & \sum_{j_1=-\infty}^0 \sum_{j_2=-\infty}^0 \sum_{j_3=-\infty}^0 \lambda_1^{-j_1} \lambda_2^{-j_2} \lambda_3^{-j_3} \lambda_4^{S-j_1-j_2-j_3} \\ &= \frac{\lambda_4^S}{(1-\lambda_4\lambda_1)(1-\lambda_4\lambda_2)(1-\lambda_4\lambda_3)} \end{aligned}$$

Adding the eight results and subtracting $F_3(\lambda_2, \lambda_3, \lambda_4; S) + F_3(\lambda_1, \lambda_3, \lambda_4; S) + F_3(\lambda_1, \lambda_2, \lambda_4; S)$ to remove the duplicate contribution of the three coordinate planes; further subtracting $F_2(\lambda_1, \lambda_4; S) + F_2(\lambda_2, \lambda_4; S) + F_2(\lambda_3, \lambda_4; S)$ to remove the originally quadruple (now duplicate) contribution of the three axes; and finally subtracting λ_4^S to remove the remaining, originally eightfold (now duplicate) contribution of the origin, yields the final formula for $F_4(\lambda_1, \lambda_2, \lambda_3, \lambda_4; S)$. Not surprisingly, it turns out to be equal to (9) with $\ell = 4$.

2.3 Further challenge

At this point, it is fairly obvious that $F_5(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5; S)$ will be given by (9) with $\ell = 5$, etc. To prove this by the technique of this article becomes increasingly more difficult (impossible in general, since ℓ can have any integer value). One clearly needs to proceed by induction - would anyone want to try?

References

- [1] VRBIK Jan: “Moments of AR(k) parameter estimators” *Communications in Statistics - Simulation and Computation* **44** (2015) 1239-1252
- [2] LIU Yuhao: “Finding moments of AR(k)-model parameter estimators” *Brock Reports in Mathematics and Statistics No. 150504* (May 4, 2015)